Exercises for the “Neutrino mass models” lectures

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$$\Delta = \begin{pmatrix} \Delta^{++} \\ \Delta^{+} \\ \Delta^{0} \end{pmatrix} . \tag{1}$$

Show that a non-zero $\langle \Delta^0 \rangle$ implies a deviation from $\rho = m_W^2/(m_Z^2 \cos^2\theta_W) = 1$. Find an upper limit for $\langle \Delta^0 \rangle$ if the deviations from $\rho = 1$ are constrained to be below $\sim 10^{-3}$.

3 Exercises for lecture 3

Exercise 3.1 Lepton flavor violation. A typical signature in neutrino mass models is lepton flavor violation, with sizable rates for processes such as $\mu \to e\gamma$. However, in the simplest model with Dirac neutrino masses the smallness of neutrino masses implies tiny rates. Estimate the suppression factor and convince yourself that the resulting rate would be too small to be observable. Can you imagine a way out? Hint: keep in mind that the PMNS matrix is unitary if the light neutrinos do not mix with additional states.

4 Solutions

Exercises for lecture 1

Exercise 1.1 Show that a Dirac fermion is equivalent to two mass-degenerate Majorana fermions.

The first step is to prove two important identities with fermion fields:

$$\overline{\psi^c}\psi^c = \overline{\psi}\psi\,, \tag{2}$$

$$\overline{\psi^c}\partial\!\!\!/\psi^c = \overline{\psi}\partial\!\!\!/\psi\,. \tag{3}$$

In order to prove them, first we must recall the definition of the charge-conjugated field, $\psi^c \equiv C\,\overline{\psi}^T$, which in turn implies $\overline{\psi^c} = \psi^T C$. Then, one finds

$$\overline{\psi^c}\psi^c = \psi^T C C\,\overline{\psi}^T = -\psi^T\overline{\psi}^T = \overline{\psi}\psi\,, \tag{4}$$

where we have used $C^2 = -1$ and the fact that fermion fields are anti-commuting Grassman variables following $(ab)^T = -b^T a^T$. A global transposition can be removed, as it has no effect. Similarly, we obtain

$$\overline{\psi^c}\partial\!\!\!/\psi^c = \psi^T C\gamma_\mu\partial^\mu C\,\overline{\psi}^T = \psi^T\gamma_\mu^T\partial^\mu\overline{\psi}^T = -\left(\partial^\mu\overline{\psi}\right)\gamma_\mu\psi = -\partial^\mu\left(\overline{\psi}\gamma_\mu\psi\right) + \overline{\psi}\gamma_\mu\partial^\mu\psi\,, \tag{5}$$

where we have used $C\gamma_\mu C = \gamma_\mu^T$ and, in the third equality, again the fact that fermion fields introduce a negative sign when they are reordered. We see that dropping a total derivative, of no physical effect, we recover Eq. (3). Therefore, we have proved the two identities in Eqs. (2) and (3). These now allow us to rewrite the Dirac Lagrangian as

$$\mathcal{L}_D = i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi = \frac{i}{2}\left(\overline{\psi}\partial\!\!\!/\psi + \overline{\psi^c}\partial\!\!\!/\psi^c\right) - \frac{m}{2}\left(\overline{\psi}\psi + \overline{\psi^c}\psi^c\right)\,. \tag{6}$$

Now we can define the two Majorana fields

$$\psi_M^1 = \frac{1}{\sqrt{2}}\left(\psi + \psi^c\right)\,, \tag{7}$$

$$\psi_M^2 = \frac{i}{\sqrt{2}}\left(\psi - \psi^c\right)\,, \tag{8}$$

which clearly fulfill the Majorana condition, $\left(\psi_M^1\right)^c = \psi_M^1$ and $\left(\psi_M^2\right)^c = \psi_M^2$. Equations (7) and (8) are equivalent to

$$\psi = \frac{1}{\sqrt{2}}\left(\psi_M^1 - i\,\psi_M^2\right)\,, \tag{9}$$

$$\psi^c = \frac{1}{\sqrt{2}}\left(\psi_M^1 + i\,\psi_M^2\right)\,, \tag{10}$$

and

$$\overline{\psi} = \frac{1}{\sqrt{2}}\left(\overline{\psi_M^1} + i\,\overline{\psi_M^2}\right)\,, \tag{11}$$

$$\overline{\psi^c} = \frac{1}{\sqrt{2}}\left(\overline{\psi_M^1} - i\,\overline{\psi_M^2}\right)\,. \tag{12}$$

Replacing Eqs. (9)-(12) into the Lagrangian in Eq. (6), one easily finds

$$\mathcal{L}_D = \frac{i}{2}\left(\overline{\psi_M^1}\partial\!\!\!/\psi_M^1 + \overline{\psi_M^2}\partial\!\!\!/\psi_M^2\right) - \frac{m}{2}\left(\overline{\psi_M^1}\psi_M^1 + \overline{\psi_M^2}\psi_M^2\right)\,, \tag{13}$$

which is the sum of two Majorana Lagrangians for the mass-degenerate fields ψ_M^1 and ψ_M^2.

Exercises for lecture 2

Exercise 2.1 Is it possible to construct a realistic model that generates neutrino masses at the 5-loop level?

In order to answer this question, let us take the master formula for radiatively generated Majorana neutrino masses. This reads

$$m_\nu \sim \frac{\langle \Phi \rangle^2}{\Lambda} \times \left(\frac{1}{16\pi^2} \right)^n , \tag{14}$$

where n is the loop level at which neutrino masses are generated. Notice that this expression assumes that all couplings are of order ~ 1, and therefore they cannot be made larger in order to increase the resulting neutrino mass. [1] Taking $\langle \Phi \rangle = v/\sqrt{2}$, with $v = 246$ GeV, one can find the required high-energy scale Λ to obtain $m_\nu \sim 0.1$ eV for different values of n. This is the result:

n	Λ [GeV]
0	$3 \cdot 10^{14}$
1	10^{11}
2	$5 \cdot 10^{7}$
3	$2 \cdot 10^{5}$
4	7
5	$3 \cdot 10^{-3}$

We see that the case $n = 4$ is already in tension with the fact that Λ must be clearly above the electroweak scale for the effective field theory description to make sense. This can be cured, however, by assuming an enhancement factor, possibly coming from a large multiplicity in the loop. For instance, this is the trick used in the 4-loop neutrino mass model in [3]. In this model the diagram leading to neutrino masses contains a N_B^6 factor, where N_B is the number of generations of charged scalars. This potentially large factor can easily enhance the resulting neutrino mass and compensate for the 4-loop suppression.

However, it seems impossible to reconcile $n = 5$ with $\Lambda \gg v$, since the enhancement factor would be very large in this case. One would need a huge multiplicity in the loop, which clearly seems artificial. Therefore, we conclude that it appears rather unlikely that a realistic 5-loop neutrino mass model can be built.

Exercise 2.2 Consider the SM extended with a real scalar $\boldsymbol{\Delta}$ with quantum numbers $(\mathbf{1},\mathbf{3})_\mathbf{1}$ under the SM gauge group and decomposed in $SU(2)_L$ components as

$$\boldsymbol{\Delta} = \begin{pmatrix} \boldsymbol{\Delta}^{++} \\ \boldsymbol{\Delta}^{+} \\ \boldsymbol{\Delta}^{0} \end{pmatrix} .$$

Show that $\langle \boldsymbol{\Delta}^0 \rangle \neq 0$ implies a deviation from $\rho = m_W^2/(m_Z^2 \cos^2\theta_W) = 1$. Find an upper limit for $\langle \boldsymbol{\Delta}^0 \rangle$ if the deviations from $\rho = 1$ are constrained to be below $\sim 10^{-3}$.

The covariant derivative for the scalar triplet Δ is

$$D_\mu \Delta = \left(\partial_\mu - ig \vec{T}\vec{W}_\mu - ig' B_\mu \right) \Delta , \tag{15}$$

since $Y_\Delta = 1$. Here T_i, $i = 1, 2, 3$, are the $SU(2)$ generators in the triplet representation. These are given by

$$T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad T_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad , \quad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{16}$$

[1] If some of the couplings entering the expression for neutrino masses are small, a new suppression would be present. In this case lower Λ scales would be required to accommodate a neutrino mass in the 0.1 eV ballpark, thus making the following argument even stronger.

In order to find the Δ contribution to the gauge boson masses we must compute

$$\left(D_\mu \langle\Delta\rangle\right)^\dagger D^\mu \langle\Delta\rangle \,. \tag{17}$$

The Δ VEV, $\langle\Delta\rangle$, is

$$\langle\Delta\rangle = \begin{pmatrix} 0 \\ 0 \\ v_\Delta \end{pmatrix} , \tag{18}$$

so that $v_\Delta \neq 0$ is the Δ^0 VEV. We note that this is the only direction in which Δ can get a VEV, since otherwise electric charge would be spontaneously broken:

$$\Delta = \begin{pmatrix} \Delta^{++} \\ \Delta^{+} \\ \Delta^{0} \end{pmatrix} \quad \begin{matrix} \longleftarrow & Q = T_3 + Y = +2 \\ \longleftarrow & Q = T_3 + Y = +1 \\ \longleftarrow & Q = T_3 + Y = \quad 0 \end{matrix}$$

Direct computation gives

$$\begin{aligned} \left(D_\mu \langle\Delta\rangle\right)^\dagger D^\mu \langle\Delta\rangle &= g^2 v_\Delta^2 W_\mu^+ W^{-\mu} + v_\Delta^2 \left(g' B_\mu - g W_\mu^3\right) \left(g' B^\mu - g W^{3\mu}\right) \\ &= g^2 v_\Delta^2 W_\mu^+ W^{-\mu} + \left(g^2 + g'^2\right) v_\Delta^2 Z_\mu Z^\mu \,. \end{aligned} \tag{19}$$

In the last step we have used that $\left(g' B_\mu - g W_\mu^3\right) \left(g' B^\mu - g W^{3\mu}\right) = \left(g^2 + g'^2\right) Z_\mu Z^\mu$, as can be easily found just by replacing B_μ and W_μ^3 in terms of A_μ and Z_μ. Therefore, v_Δ contributes to the $W^\pm$ and Z masses. Adding the usual SM contributions, we find [2]

$$m_W^2 = \frac{g^2 v^2}{4} + g^2 v_\Delta^2 = \frac{g^2 v^2}{4} \left(1 + \frac{4 v_\Delta^2}{v^2}\right) , \tag{20}$$

$$m_Z^2 = \frac{v^2}{4} \left(g^2 + g'^2\right) + 2\, v_\Delta^2 \left(g^2 + g'^2\right) = \frac{v^2}{4} \left(g^2 + g'^2\right) \left(1 + \frac{8 v_\Delta^2}{v^2}\right) , \tag{21}$$

where $\langle\Phi\rangle = v/\sqrt{2}$ is the Higgs boson VEV. Finally, we obtain

$$\rho = \frac{m_W^2}{\cos^2\theta_W m_Z^2} = \frac{1 + \frac{4 v_\Delta^2}{v^2}}{1 + \frac{8 v_\Delta^2}{v^2}} , \tag{22}$$

where we have used that $\cos^2\theta_W = g^2 / \left(g^2 + g'^2\right)$. Since ρ is experimentally known to be very close to 1, the result in Eq. (22) implies $v_\Delta \ll v$. More precisely, imposing that the deviation from $\rho = 1$ is below the $\sim 10^{-3}$ level, $1 - \rho \lesssim 10^{-3}$, we find

$$v_\Delta \lesssim 4\,\text{GeV} \,. \tag{23}$$

[2]Keep in mind that the Z boson mass term contains a factor of $\frac{1}{2}$: $\mathcal{L}_Z^{\text{mass}} = \frac{1}{2} m_Z^2 Z_\mu Z^\mu$.

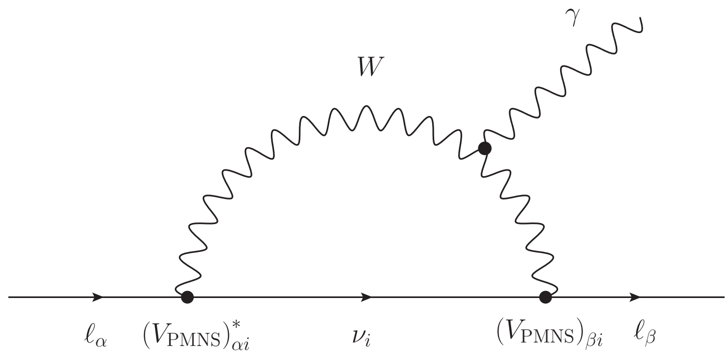

Figure 1: Feynman diagram for the decay $\ell_\alpha \to \ell_\beta \gamma$ in the simplest Dirac neutrino mass model.

Exercises for lecture 3

Exercise 3.1 A typical signature in neutrino mass models is lepton flavor violation, with sizable rates for processes such as $\mu \to e\gamma$. However, in the simplest model with Dirac neutrino masses the smallness of neutrino masses implies tiny rates. Estimate the suppression factor and convince yourself that the resulting rate would be too small to be observable. Can you imagine a way out? Hint: keep in mind that the PMNS matrix is unitary if the light neutrinos do not mix with additional states.

In the simplest Dirac neutrino mass model, all processes with lepton flavor violation (LFV) are induced at loop level. For instance, Fig. 1 shows the leading Feynman diagram for the decay $\ell_\alpha \to \ell_\beta \gamma$. Notice the presence of a PMNS matrix element in each vertex. The amplitude for this process takes the generic form

$$\mathcal{M} \sim g^2 \sum_i (V_{\mathrm{PMNS}})^*_{\alpha i} (V_{\mathrm{PMNS}})_{\beta i} \, F(x_i) \, , \tag{24}$$

where $F(x_i)$ is a loop function that depends on $x_i = (m_{\nu_i}/m_W)^2$. We can now expand F in powers of the (very small) quantity $(m_{\nu_i}/m_W)^2$. We obtain

$$F(x_i) = a_0 + a_2 \left(\frac{m_{\nu_i}}{m_W} \right)^2 + \ldots \tag{25}$$

with a_0 and a_2 two numerical coefficients. Higher orders in $(m_{\nu_i}/m_W)^2$ can be safely neglected. Replacing this result in Eq.(24) one finds

$$\mathcal{M} \sim g^2 \sum_i (V_{\mathrm{PMNS}})^*_{\alpha i} (V_{\mathrm{PMNS}})_{\beta i} \left[a_0 + a_2 \left(\frac{m_{\nu_i}}{m_W} \right)^2 \right] = g^2 a_2 \sum_i (V_{\mathrm{PMNS}})^*_{\alpha i} (V_{\mathrm{PMNS}})_{\beta i} \left(\frac{m_{\nu_i}}{m_W} \right)^2 \, , \tag{26}$$

where in the last step we have used that (in the model under consideration) the PMNS matrix is unitary, [3]

$$\sum_i (V_{\mathrm{PMNS}})^*_{\alpha i} (V_{\mathrm{PMNS}})_{\beta i} = \left(V_{\mathrm{PMNS}} V^\dagger_{\mathrm{PMNS}} \right)_{\beta\alpha} = \mathbb{I}_{\beta\alpha} \quad (= 0 \ \text{for} \ \alpha \neq \beta) \, . \tag{27}$$

The result in Eq.(26) must be squared to compute the $\ell_\alpha \to \ell_\beta \gamma$ rate, which is then suppressed by the factor $(m_{\nu_i}/m_W)^4$. [4] For $m_\nu \lesssim 0.1$ eV, this suppression factor is of the order of $\sim 10^{-48}$. With such a strong suppression the resulting rate would be too small to be observed. Notice that the current best experimental limit for the branching ratio of a $\ell_\alpha \to \ell_\beta \gamma$ decay is for the $\mu \to e\gamma$ case and is $4.2 \cdot 10^{-13}$ [2].

[3] This is just another incarnation of the GIM mechanism [1]. The unitarity of the mixing matrix entering the amplitude leads to a complete cancellation of the leading term, resulting in a strong suppression due to the smallness of the masses involved in the problem.

[4] More precisely, the factor is actually $(\Delta m_\nu/m_W)^4$, with Δm_ν a mass difference between two neutrino mass eigenstates. Note that if the light neutrinos were completely degenerate, one could factor out $(m_\nu/m_W)^2$ from the amplitude and find a vanishing result again due to $\sum_i (V_{\mathrm{PMNS}})^*_{\alpha i} (V_{\mathrm{PMNS}})_{\beta i} = 0$. However, we can assume that neutrino masses and neutrino mass differences are of similar sizes and ignore this detail for the estimate of the suppression factor.

Is there a way out of this prediction? Of course there is. The result we have obtained depends on two assumptions (valid in the minimal Dirac model we have considered): (i) a unitary PMNS matrix, and (ii) the neutrinos running in the loop are the standard light neutrinos. It is very easy to construct models where these assumptions (or at least one of them) are violated. For instance, in models in which the Standard Model neutrinos mix additional singlet fermions (such as type-I seesaw-like models), the 3×3 PMNS matrix in the light neutrino sector is not unitary. The additional singlet fermions can also run in the loop, completely changing the relevant mass ratio. And finally, there is a vast amount of models sources of LFV not necessarily related in a direct way to neutrino masses. In these scenarios there is no suppression due to neutrino masses.

References

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[3] T. Nomura and H. Okada, "A four-loop Radiative Seesaw Model," *Phys. Lett.* **B770** (2017) 307–313, `arXiv:1601.04516 [hep-ph]`.