2 Lecture 1: Towards the Standard Model

2.1 Weak phenomena and first theories for the weak interactions

We will begin by discussing the first steps towards the SM, given in the first half of the last century with the proposal of several theories for the weak interactions. As we will see, these were based on crucial experimental discoveries which served as guiding tools towards the correct theoretical ideas.

Experimental facts

Several important discoveries pawed the way for the construction of a theory for the weak interactions.

β-decay The discovery of radioactivity by Becquerel in 1896 can be regarded as the discovery (or at least the first step towards) the weak interactions. Later, in 1914, Chardwick showed that the electrons produced in β-decay have a continuous spectrum, a fact that was explained by Pauli in 1930 with the neutrino hypothesis [7]. In 1934, Fermi published a landmark theory for β-decay [8,9], based on his famous 4-fermion interaction Lagrangian

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}} \overline{p} \gamma_\mu n \, \overline{e} \gamma^\mu \nu + \text{h.c.} \, , \tag{1}$$

where $G_F = 1.166 \cdot 10^{-5}$ GeV^{-2} is the so-called Fermi constant.

μ and π decays Muons and pions were found in cosmic rays experiments in the 30's and 40's, respectively. Among other interesting properties, they were found to have, like β-decays, comparatively long lifetimes,

$$\tau(\mu^\pm) = 2.2 \cdot 10^{-6} \, \text{s} \quad , \quad \tau(\pi^\pm) = 2.6 \cdot 10^{-8} \, \text{s} \, . \tag{2}$$

Even though these lifetimes may seem quite short, they are in fact orders of magnitude longer than the typical lifetime for strong decays, $\tau_s \sim 10^{-23}$ s. The fact that a number of particle decays had similarly long lifetimes was a remarkable observation. Eventually, the concept of a distinctive class of interactions, the "weak interactions", began to emerge. These were characterized by being short-range and much weaker than the electromagnetic and strong interactions.

Lepton number and lepton flavor conservation It was also observed that the neutrinos associated to the electron and the muon conserved *flavors* and thus were two different particles. While processes like

$$\nu_\mu X \to \mu^- X' \tag{3}$$

are possible, the analogous

$$\nu_\mu X \to e^- X' \tag{4}$$

are not (here X and X' are not leptons). Similarly, the number of leptons was found to be a conserved quantity.

Parity violation Several experimental results showed that the weak interactions violate parity. First, two different decays were found for charged strange mesons

$$\theta^+ \to \pi^+ \pi^0 \, , \tag{5}$$
$$\tau^+ \to \pi^+ \pi^+ \pi^- \, . \tag{6}$$

Since the intrinsic parity of a pion is $P_\pi = -1$, the two final states have different parity. For this reason, it was initially thought that the θ and τ mesons were two different particles. However, increasingly precise experiments were unable to find any difference between their masses and lifetimes, suggesting that they were the same particle. This was the so-called $\theta - \tau$ puzzle. Nowadays we know that this particle is the K^+ meson, which can decay violating parity via the weak interactions, as originally suggested by Lee and Yang in 1956 as possible solution to the puzzle [10]. The confirmation of this idea came one year later, in 1957, with the celebrated discovery by Wu of parity violation in the β-decay of Cobalt-60 nuclei [11] [1]. This indicates that a theory based on 4-fermion contact interactions, such as Fermi's, should contain γ_5 matrices in the interaction terms, as these distinguish between left (L) and right (R) fermion chiralities.

[1]Other experiments performed in the next couple of years confirmed this result and showed that the electrons emitted in weak interactions are mostly left-handed, clearly violating parity.

Strangeness violating decays Decays with violation of strangeness, such as

$$K \to \pi \ell^- \overline{\nu}_\ell \,, \tag{7}$$

$$\Lambda \to p e^- \overline{\nu}_e \,, \tag{8}$$

present two remarkable features: (i) the strength is the same in all decays but smaller than in the $\Delta S = 0$ processes (like $\pi \to \mu^- \overline{\nu}_\mu$), with $G_{\Delta S=1} \simeq 0.22\, G_F$, and (ii) all decays satisfy the $\Delta S = \Delta Q$ rule (in the hadronic part of the decay), so that processes like $\Sigma^+ \to n e^+ \nu_e$ never occur.

The V-A theory

It was soon realized by several theorists (Feynman, Gell-Mann, Sudarshan, Marshak, Sakurai ...) that all previous experimental facts can be described by

$$\mathcal{L}_{\text{V-A}} = -\frac{G_F}{\sqrt{2}}\, J_\alpha^\dagger J^\alpha + \text{h.c.}\,, \tag{9}$$

with the weak current J^α being of the vector-minus-axial (V-A) form. This is the important **V-A theory** for the weak interactions.

More explicitly, J^α can be split into leptonic and hadronic parts,

$$J^\alpha = J_\ell^\alpha + J_h^\alpha \,. \tag{10}$$

Therefore, the $\mathcal{L}_{\text{V-A}}$ Lagrangian can be used to describe leptonic, semi-leptonic and purely hadronic processes. The leptonic current is simply given by

$$J_\ell^\alpha = \overline{\nu}_e \gamma^\alpha (1-\gamma_5) e + \overline{\nu}_\mu \gamma^\alpha (1-\gamma_5) \mu \,, \tag{11}$$

whereas the hadronic current is written in terms of quarks as

$$J_q^\alpha = \overline{u} \gamma^\alpha (1-\gamma_5) \left(\cos\theta_c d + \sin\theta_c s \right) \,. \tag{12}$$

Here θ_c is the Cabibbo angle, introduced by Cabibbo to recover the universality of the weak interactions in $\Delta S = 1$ transitions [12]. In fact, one finds

$$G_{\Delta S=0} = \cos\theta_c \, G_F \simeq 0.97\, G_F \tag{13}$$

$$G_{\Delta S=1} = \sin\theta_c \, G_F \simeq 0.22\, G_F \tag{14}$$

since $\sin\theta_c$ is measured to be about 0.22.

There are a couple of properties of the V-A theory worth emphasizing:

- It only involves left-handed fermions:

Using the standard definition of the chirality projectors $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$, one finds

$$\overline{\psi} \gamma_\mu (1-\gamma_5) \psi \equiv 2\, \overline{\psi}_L \gamma_\mu \psi_L \,. \tag{15}$$

explicitly showing that only left-handed fermions participate in the weak interactions. This is equivalent to the well-known rule that "parity is maximally violated in the weak interactions", since only one of the chiralities takes of part of them.

- It is written in terms of charged currents, all with one unit of charge. In fact, in the lowest order in perturbation theory (tree-level) there are no neutral current processes such as $\nu_\mu X \to \nu_\mu X$.

If we restrict the application of the V-A theory to the leading order in G_F, it is able to describe correctly a vast amount of low-energy weak-interaction data involving processes of many types:

- β-decay (also inverse): $n \to p e^- \overline{\nu}_e$, $e^- p \to \nu_e n$.
- μ, τ decays: $\mu^- \to e^- \overline{\nu}_e \nu_\mu$, $\tau^- \to \mu^- \overline{\nu}_\mu \nu_\tau$, $\tau^- \to \nu_\tau \pi^-$, ...
- π, K decays: $\pi^+ \to \mu^+ \nu_\mu$, $\pi^+ \to \pi^0 e^+ \nu_e$, $K^+ \pi^+ \pi^0$, ...
- Hyperon decays: $\Lambda \to p \pi^-$, ...
- ν scattering: $\nu_\mu e^- \to \mu^- \nu_e$, ...

However, the V-A theory cannot be considered as a consistent QFT of the weak interactions. There are two reasons for this:

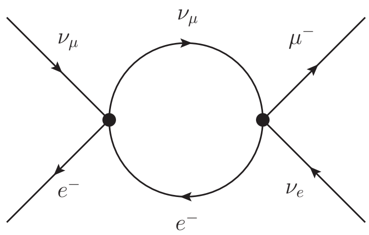

Figure 1: 1-loop contribution to inverse muon decay in the V-A theory.

1) Violation of unitarity The coupling constant G_F has dimension $(\text{mass})^{-2}$. Any amplitude that contains one 4-fermion interaction will be proportional to G_F, so cross-sections will be proportional to G_F^2, with dimension $(\text{mass})^{-4}$. Since these must have units of $(\text{mass})^{-2}$, the dimensions must be compensated by the square of the characteristic energy in the process. In short, dimensional analysis tells us that

$$\sigma \sim G_F^2 s \,. \tag{16}$$

This is not acceptable for arbitrarily large s, since the cross-section is bounded by the fact that the probability of any two particles to scatter cannot exceed 1. If it grows with energy, eventually this bound, known as unitarity bound, will be violated.

Indeed, the V-A theory violates unitarity at relatively low energies. For instance, inverse muon decay,

$$\nu_\mu e^- \to \mu^- \nu_e \,, \tag{17}$$

violates unitarity at $\sqrt{s} \sim 300$ GeV.

2) Lack of renormalizability One may hope that unitarity is restored by including higher (loop) corrections, such as that shown in Fig. 1. However, this is not the case. In fact, higher-order contributions are increasingly divergent, with infinities that cannot be absorbed in the parameters of the model. As a consequence of this, the V-A theory is not renormalizable and cannot be used beyond leading order. We note that this is again related to the Fermi constant's dimensionality, which is $(\text{mass})^{-2}$.

Therefore, even though the V-A theory can correctly account for a large domain of weak phenomena, it cannot be "the theory" of weak interactions.

The intermediate vector boson (IVB) theory

In Quantum ElectroDynamics (QED), the fundamental $\gamma \bar{e} e$ interaction generates a long-range 4-fermion interaction through γ-exchange. One can try to generate 4-fermion weak interactions in a similar way.

This idea was first put forward by Schwinger, and independently by Lee and Yang, who introduced the so-called intermediate vector boson (IVB) theory. In this case the Lagrangian is given by

$$\mathcal{L}_{\text{IVB}} = \frac{g}{2\sqrt{2}} \left(J^\mu W_\mu^+ + \text{h.c.} \right) \,, \tag{18}$$

where W_μ is a new massive (since the weak interactions are short-range) charged spin-1 field.

The V-A theory can be seen as the low-energy limit of the IVB theory. This can be easily understood by looking at the amplitudes for a given 4-fermion process in both theories. Let us consider, for instance, muon decay. In this case one gets

$$\mathcal{M}_{\text{V-A}} = \frac{G_F}{\sqrt{2}} \, J^{\dagger\alpha}(\mu) J_\alpha(e) \,, \tag{19}$$

$$\mathcal{M}_{\text{IVB}} = \frac{g}{2\sqrt{2}} \, J^{\dagger\alpha}(\mu) \, \frac{-g_{\alpha\beta} + \frac{q_\alpha q_\beta}{m_W^2}}{q^2 - m_W^2} \, \frac{g}{2\sqrt{2}} \, J^{\dagger\beta}(e) \,, \tag{20}$$

where $J^\alpha(e,\mu)$ are the leptonic currents and q is the 4-momentum of the W boson exchanged between th two currents in the IVB theory. At low energies, $q^2 \ll m_W^2$, the W boson propagator becomes

$$\frac{-g_{\alpha\beta} + \frac{q_\alpha q_\beta}{m_W^2}}{q^2 - m_W^2} \quad \xrightarrow{q^2 \ll m_W^2} \quad \frac{g_{\alpha\beta}}{m_W^2} \,, \tag{21}$$

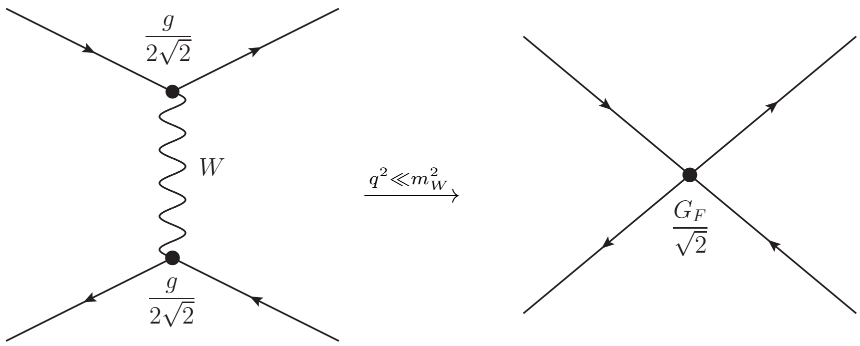

Figure 2: Muon decay in the V-A and IVB theories. The diagram on the right (V-A theory) is obtained by collapsing the W propagator in the diagram on the left (IVB theory) to a point.

and therefore

$$\mathcal{M}_{\mathrm{IVB}} \quad \xrightarrow{q^2 \ll m_W^2} \quad \frac{g^2}{8m_W^2} J^{\dagger\alpha}(\mu) J_\alpha(e)\,, \tag{22}$$

which then allows us to identify

$$\frac{g^2}{8m_W^2} = \frac{G_F}{\sqrt{2}}\,. \tag{23}$$

Graphically, this can be represented by the collapse of the W propagator to a point, leading to the well-known amplitude, as shown in Fig. 2.

Let us now consider the problems of the V-A theory. Are they solved in the IVB theory?

1. The problem with unitarity remains. Although the behavior in $\nu_\mu e^- \to \mu^- \nu_e$ clearly improves and unitarity is preserved up to very high energies ($\sim 10^{15}$ GeV), the problem appears for example in the reaction $\nu\bar{\nu} \to W_L^+ W_L^-$, where W_L denotes a longitudinally polarized W boson. Again, unitarity is violated at quite low energies, as in the V-A theory.

2. Even though g is now a dimensionless coupling, the IVB theory is not renormalizable either. The problem is caused by the $q_\mu q_\nu / m_W^2$ piece in the W propagator. At high energies,

$$\frac{-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_W^2}}{q^2 - m_W^2} \quad \xrightarrow{q^2 \to \infty} \quad \text{constant}\,, \tag{24}$$

and the interaction is not renormalizable by power counting.

We conclude that although the IVB theory improves with respect to the V-A theory, the known problems still remain. One of the key issues seems to be the introduction of a W boson mass without spoiling renormalizability. As we will see, this will be possible in a gauge theory with spontaneous symmetry breaking.

2.2 Gauge theories

Symmetries play a crucial role in particle physics. We will now study one of the most important ideas that led to the development of the SM: gauge invariance.

Abelian gauge theory – QED

QED is an Abelian gauge theory. It is instructive to see how the theory can be derived by requiring the Dirac free electron theory to be invariant under local transformations.

Let us consider the Lagrangian for a free electron

$$\mathcal{L}_0 = \overline{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi(x)\,. \tag{25}$$

It has a global $U(1)$ symmetry due to the invariance of $\mathcal{L}_0$ under a phase transformation,

$$\psi(x) \quad \to \quad \psi'(x) = e^{-i\alpha}\psi(x) \tag{26}$$

$$\overline{\psi}(x) \quad \to \quad \overline{\psi}'(x) = e^{i\alpha}\overline{\psi}(x)\,. \tag{27}$$

This transformation is said to be "global", since the field is transformed exactly in the same way in the whole universe. It makes sense to think that fundamental symmetries should be "local" (or "gauge"), with parameters depending on the coordinates. This is the gauge principle [13]. Therefore, let us gauge the theory replacing α by $\alpha(x)$,

$$\psi(x) \quad \to \quad \psi'(x) = e^{-i\alpha(x)}\psi(x) \tag{28}$$

$$\overline{\psi}(x) \quad \to \quad \overline{\psi}'(x) = e^{i\alpha(x)}\overline{\psi}(x)\,. \tag{29}$$

It is obvious that the derivative term spoils the invariance. In fact, we find

$$\begin{aligned}\overline{\psi}(x)\partial_\mu\psi(x) \quad \to \quad \overline{\psi}'(x)\partial_\mu\psi'(x) &= \overline{\psi}(x)e^{i\alpha(x)}\partial_\mu\left(e^{-i\alpha(x)}\psi(x)\right) \\ &= \overline{\psi}(x)\partial_\mu\psi(x) - i\,\overline{\psi}(x)\partial_\mu\alpha(x)\,\psi(x) \\ &\neq \overline{\psi}(x)\partial_\mu\psi(x)\,.\end{aligned} \tag{30}$$

In order to recover the invariance we must replace the usual derivative ∂_μ by a covariant derivative D_μ that transforms like the field

$$D_\mu\psi(x) \quad \to \quad (D_\mu\psi(x))' = e^{-i\alpha(x)}\psi(x)\,, \tag{31}$$

so that invariance is trivially recovered

$$\overline{\psi}(x)D_\mu\psi(x) \quad \to \quad \overline{\psi}'(x)\,(D_\mu\psi(x))' = \overline{\psi}(x)D_\mu\psi(x)\,. \tag{32}$$

This can be done by enlarging the theory with a new vector field $A_\mu(x)$, the "gauge field", so that the covariant derivative is

$$D_\mu = \partial_\mu + ieA_\mu\,, \tag{33}$$

where e is a free parameter. Then, the transformation law for the covariant derivative will be satisfied if $A_\mu(x)$ has the transformation property

$$A_\mu(x) \quad \to \quad A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)\,. \tag{34}$$

With this, we have found that the Lagrangian

$$\mathcal{L}'_0 = \overline{\psi}i\gamma^\mu\left(\partial_\mu + ieA_\mu\right)\psi - m\overline{\psi}\psi \tag{35}$$

is invariant under local $U(1)$ transformations. However, to make the gauge field a truly dynamical variable, we must add a kinetic term for A_μ, this is, a term involving its derivatives. It must be quadratic in the field and gauge invariant. The only term one can build with these properties is

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{36}$$

where the $1/4$ factor is introduced to get the conventional normalization for the kinetic term and we have defined

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{37}$$

It is straightforward to show that $F_{\mu\nu}$ is gauge invariant by itself, and then $\mathcal{L}_A$ obviously is invariant as well. Combining all these ingredients we arrive at the celebrated QED Lagrangian

$$\mathcal{L}_{\rm QED} = \overline{\psi}i\gamma^\mu\left(\partial_\mu + ieA_\mu\right)\psi - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{38}$$

This Lagrangian is absolutely successful describing electromagnetic interactions. Let us make some brief remarks about its properties:

- It describes the interaction of a $q = -1$ particle with the photon. Generalization to a more general charge q is obtained by replacing the transformation in Eqs. (28) and (29) by $\psi \to \exp\left(iq\alpha(x)\right)\psi$.
- The photon (A_μ) is massless because a $A_\mu A^\mu$ term is not gauge invariant.
- The minimal coupling of the photon is contained in the covariant derivative $D_\mu\psi$. In other words: the interactions are determined by the gauge symmetry.
- The gauge field does not have self-interactions.

$U(1)$ is an Abelian symmetry group: rephasings conmute. We will now see how the gauge principle can be applied in the non-Abelian case.

Non-Abelian gauge theories – Yang-Mills theories

In 1954 Yang and Mills extended the gauge principle to non-Abelian symmetry groups [14]. We will now illustrate this for $SU(2)$, the group of 2×2 unitary matrices with determinant equal to 1.

A detailed (and rigurous) treatment of Lie groups and their application to particle physics is clearly beyond the scope of this course. For this reason, let us simply focus on their most relevant features for the construction of the SM by looking at a simple example. Consider the doublet of fermions

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \tag{39}$$

Similarly to the $U(1)$ transformations in Eqs. (28) and (29), one can define an $SU(2)$ transformation acting on ψ as

$$\psi \quad \to \quad \psi' = U(\theta)\, \psi \,, \tag{40}$$

with

$$U(\theta) = \exp\left(- \frac{i \vec{\tau} \vec{\theta}}{2} \right) . \tag{41}$$

Here $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ are the $SU(2)$ transformation parameters and $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices. Indeed, the generators of $SU(2)$ for the doublet representation are $T_i = \tau_i/2$, with $i = 1, 2, 3$. They satisfy the $SU(2)$ algebra

$$\left[\frac{\tau_i}{2}, \frac{\tau_j}{2} \right] = i\, \epsilon_{ijk} \frac{\tau_k}{2} \,, \tag{42}$$

where ϵ_{ijk} is the completely antisymmetric Levi-Civita tensor. The fact that these conmutators are not zero is what makes $SU(2)$ a non-Abelian symmetry group. After learning how ψ transforms under $SU(2)$ it is easy to show that the free Lagrangian

$$\mathcal{L}_0 = \overline{\psi}(x) \left(i \gamma^\mu \partial_\mu - m \right) \psi(x) \tag{43}$$

is invariant under a global $SU(2)$ transformation in which the θ_i parameters are constants. However, when $\theta_i = \theta_i(x)$, one finds that $\mathcal{L}_0$ is no longer invariant, again due to the derivative term:

$$\begin{aligned} \overline{\psi}(x) \partial_\mu \psi(x) \quad \to \quad \overline{\psi}'(x) \partial_\mu \psi'(x) &= \overline{\psi}(x) U^{-1}(\theta) \partial_\mu \left(U(\theta) \psi(x) \right) \\ &\neq \overline{\psi}(x) \partial_\mu \psi(x) \,. \end{aligned} \tag{44}$$

To construct a gauge invariant Lagrangian we follow the same procedure as for the Abelian case. We replace the derivative ∂_μ by a covariant derivative D_μ, defined as

$$D_\mu = \partial_\mu - i g \frac{\vec{\tau} \vec{A}_\mu}{2} \,, \tag{45}$$

where g is a coupling constant and we have introduced the vector bosons A_μ^i, $i = 1, 2, 3$, one for each group generator. We now demand that $D_\mu \psi$ transforms in the same way ψ does

$$D_\mu \psi(x) \quad \to \quad \left(D_\mu \psi(x) \right)' = U(\theta) \psi(x) \,, \tag{46}$$

which implies

$$\left(\partial_\mu - i\, g \frac{\vec{\tau} \vec{A}_\mu'}{2} \right) \left(U(\theta) \psi \right) = U(\theta) \left(\partial_\mu - i\, g \frac{\vec{\tau} \vec{A}_\mu}{2} \right) \psi \,, \tag{47}$$

or, equivalently

$$\frac{\vec{\tau} \vec{A}_\mu'}{2} = U(\theta) \left[\frac{\vec{\tau} \vec{A}_\mu}{2} + \frac{i}{g} \partial_\mu \right] U^{-1}(\theta) \,, \tag{48}$$

which defines the transformation law for the gauge fields. For an infinitesimal change $\theta_i \ll 1$ one can easily solve this relation and find

$$A_\mu^i \quad \to \quad A_\mu^{i\,\prime} = A_\mu^i + \epsilon^{ijk} \theta^j A_\mu^k - \frac{1}{g} \partial_\mu \theta^i \,. \tag{49}$$

The second term is new when we compare to the results obtained for the Abelian case. In fact, this term is the transformation of an $SU(2)$ triplet (the adjoint representation of $SU(2)$). Thus, we see that in contrast to the Abelian case, the A_μ^i gauge fields are charged under the symmetry group. As we will find below, this will necessarily imply that the A_μ^i vectors have self-interactions.

We must now find the correct kinetic term for the gauge fields. This can be done by generalizing the strength tensor $F_{\mu\nu}$ of the $U(1)$ interactions. In that simple case it is easy to check that

$$(D_\mu D_\nu - D_\nu D_\mu)\,\psi = ieF_{\mu\nu}\psi\,. \tag{50}$$

Therefore, we can generalize it to the non-Abelian case as

$$(D_\mu D_\nu - D_\nu D_\mu)\,\psi = ig\frac{\tau_i}{2}F^i_{\mu\nu}\psi\,. \tag{51}$$

Expanding this expression one finds

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^j_\mu A^k_\nu\,. \tag{52}$$

Now, even though $F^i_{\mu\nu}$ is not gauge invariant, the combination $F^i_{\mu\nu}F_i^{\mu\nu}$ is. We can then summarize the above discussion. The complete gauge invariant Lagrangian that describes the interaction of the A^i_μ gauge fields with the ψ $SU(2)$ doublet is

$$\mathcal{L}_{\rm YM} = \overline{\psi}i\gamma^\mu D_\mu\psi - m\overline{\psi}\psi - \frac{1}{4}F^i_{\mu\nu}F_i^{\mu\nu}\,. \tag{53}$$

This is the Yang-Mills Lagrangian for $SU(2)$. Let us make some remarks:

- The A^i_μ gauge fields are massless, as in the Abelian case, because their mass terms would break the $SU(2)$ symmetry.
- The interactions are again dictated by the gauge principle.
- In contrast to the Abelian case, the A^i_μ fields have self-interactions. This can be seen by expanding the pure gauge term

$$-\frac{1}{4}F^i_{\mu\nu}F_i^{\mu\nu} \supset -g\epsilon^{ijk}\partial_\mu A^i_\nu A^{j\,\mu}A^{k\,\nu} - \frac{g^2}{4}\epsilon^{ijk}\epsilon^{imn}A^j_\mu A^k_\nu A^{m\,\mu}A^{n\,\nu}\,, \tag{54}$$

  which gives rise to cubic and quartic interactions.

The previous discussion can be generalized to higher groups and arbitrary representations for ψ. This is done by replacing $\tau_i/2$ by the corresponding generators T_i and ϵ_{ijk} by the corresponding structure constants f_{ijk} of the gauge group.

Previously, we had seen that the development of a theory for the weak interactions eventually led to the idea of an IVB. Could this vector be a gauge field? The main obstacle seems to be the need for a non-zero mass: while gauge fields are restricted to be massless, an IVB for the weak interactions must be massive in order to explain why these are short-range. These two conflicting facts can be consistently combined with an additional ingredient: spontaneous symmetry breaking.

2.3 Spontaneous symmetry breaking

The imposition of a gauge symmetry implies the existence of massless vector bosons. If we want to avoid this feature and obtain massive vector bosons to describe the weak interactions the symmetry must be broken somehow. We could for example add a mass term for the gauge bosons *by hand*. This type of breaking is called explicit. In addition to being quite inelegant, this solution is known to alter the high-energy behavior of the theory, again spoiling renormalizability. Therefore, we must resort to a different mechanism to break the symmetry and generate the gauge boson masses: spontaneous symmetry breaking (SSB).

SSB is a well-known phenomenom in many areas of physics. A simple system that allows for an intuitive understanding is a pencil standing on its tip. Such a system exhibits a clear axial symmetry, since rotations around the pencil axis leave it invariant. However, a pencil on its tip is not a stable minimum energy configuration. Any perturbation will eventually make the pencil fall in one specific direction. Even though all directions are completely equivalent (due to the axial symmetry), choosing one of them breaks the symmetry "spontaneously".

A similar phenomenom is observed in numerous physical systems, fully invariant under a symmetry that is not preserved by the ground state. This is the idea behind the Higgs mechanism in the SM. But before we consider SSB in gauge theories, let us see what happens when a global continuous symmetry gets spontaneously broken.

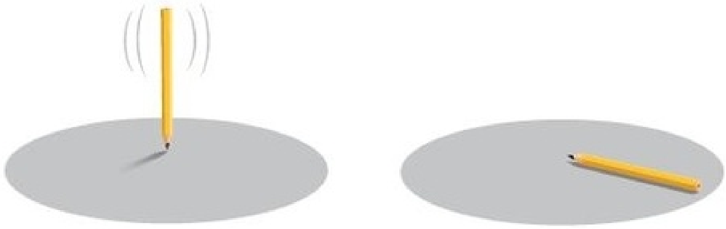

Figure 3: A pencil standing on its tip is a system with a perfect axial symmetry. All directions are completely equivalent. Eventually, the pencil falls over, choosing one specific direction. We say that the symmetry has been spontaneously broken.

SSB of a global continuous symmetry – The Goldstone theorem

Let us analyze the case of a self-interacting complex scalar field,

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi) \, , \tag{55}$$

with the scalar potential

$$V(\phi) = \mu^2 |\phi|^2 + \lambda |\phi|^4 \, . \tag{56}$$

This Lagrangian is invariant under the global transformation

$$\phi \quad \to \quad \phi' = e^{-i\theta} \phi \, . \tag{57}$$

Therefore, the system exhibits a global continuous symmetry. For instance, we note that cubic terms such as ϕ^3 are absent, as they would be forbidden by the symmetry.

Let us split the complex field into its real and imaginary parts

$$\phi = \frac{1}{\sqrt{2}} \left(\phi_1 + i \, \phi_2 \right) \, , \tag{58}$$

so that $\phi_{1,2}$ are real fields. In terms of these fields, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2 \right) - V(\phi_1, \phi_2) \, , \tag{59}$$

with

$$V(\phi_1, \phi_2) = \frac{\mu^2}{2} \left(\phi_1^2 + \phi_2^2 \right) + \frac{\lambda}{4} \left(\phi_1^2 + \phi_2^2 \right)^2 \, . \tag{60}$$

This Lagrangian is now invariant under $SO(2)$ rotations between ϕ_1 and ϕ_2,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \to \quad \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \, , \tag{61}$$

which are completely equivalent to the original rephasing transformations.

Now the question is: where is the minimum of the scalar potential? First, we note that λ should be positive to guarantee that the potential (and hence the Hamiltonian of the theory) is bounded from below. Otherwise, if $\lambda < 0$, $\phi_i \to \infty$ would lead to $V \to -\infty$, making impossible to define the ground state. For $\lambda > 0$ the location of the minimum depends on the sign of μ^2. For $\mu^2 > 0$ we just have one minimum in $\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$. Here $\langle \phi_i \rangle$ denotes the value of the scalar field ϕ_i at the minimum of the potential, also known as its vacuum expectation value (VEV). More interestingly, for $\mu^2 < 0$ ("wrong" sign for the mass term) we have a continuum of distinct minima (or "vacua") located at

$$\langle |\phi|^2 \rangle = \frac{1}{2} \left(\langle \phi_1^2 \rangle + \langle \phi_2^2 \rangle \right) = -\frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2} \, . \tag{62}$$

as shown in Fig. 4. These vacua form a circumference around the origin and thus exhibit the $SO(2)$ symmetry of the model. Now we must give an important conceptual step. In QFT we are interested in perturbations around the ground state (the vacuum), whose energy is exactly zero. Therefore, we must redefine the scalar fields in our theory in such a way that the new physical fields have vanishing VEVs. In order to do that we

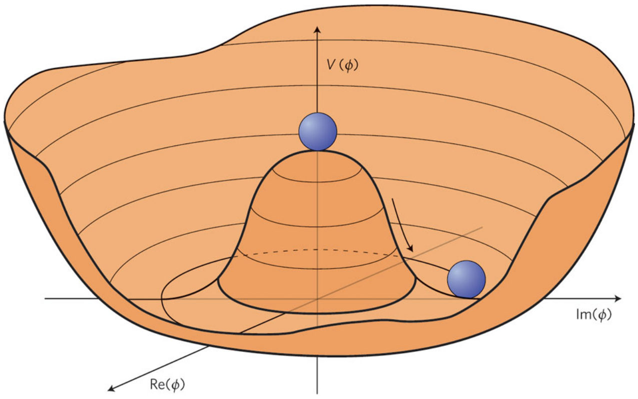

Figure 4: The scalar potential for $\mu^2 < 0$. This scalar potential is known as the "Mexican hat" potential.

must choose a specific minimum of the potential, which in turn selects a specific ground state of the theory. And this is where SSB takes place. Since the Lagrangian is invariant under $SO(2)$, all minima are equivalent. However, once the choice is made, the symmetry gets spontaneously broken since the Lagrangian is invariant but the selected vacuum ($\langle\phi\rangle$) is not.

Let us choose the minimum with

$$\langle\phi_1\rangle = v = \sqrt{\frac{-\mu^2}{\lambda}}\,, \tag{63}$$

$$\langle\phi_2\rangle = 0\,. \tag{64}$$

We define new fields, suitable for calculations in QFT,

$$\phi_1^0 = \phi_1 - v\,, \tag{65}$$

$$\phi_2^0 = \phi_2\,. \tag{66}$$

In terms of the new fields the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi_1^0\partial^\mu\phi_1^0 + \partial_\mu\phi_2^0\partial^\mu\phi_2^0\right) - \frac{1}{2}\left(-2\mu^2\right)\left(\phi_1^0\right)^2 + \text{interactions}\,. \tag{67}$$

We see that ϕ_1^0 has a real and positive mass ($-2\mu^2 > 0$), whereas ϕ_2^0 is massless since the Lagrangian does not contain any quadratic term in ϕ_2^0. Moreover, the interaction terms include cubic interactions such as $\left(\phi_1^0\right)^3$, originally forbidden.

This is an example of the Goldstone theorem [15] (1961, Goldstone), which states that when an exact continuous global symmetry is spontaneously broken, the theory contains a massless scalar particle for each broken generator of the original symmetry. These massless scalars are called Goldstone bosons.

SSB of a gauge symmetry – The Higgs mechanism

In 1964 several authors (including Guralnik, Hagen, Kibble, Higgs, Brout, Englert and others, see [16–19]) independently found a way out of the Goldstone theorem: a field theory with SSB but without Goldstone bosons. The trick consists in making the symmetry local instead of global. As a bonus, the gauge bosons become massive. This is the so-called Higgs mechanism.

In order to illustrate it let us consider an Abelian gauge theory. Let ϕ be the complex scalar field of the previous example. In order to get a Lagrangian for ϕ invariant under the local transformation

$$\phi \quad\rightarrow\quad \phi' = e^{-i\theta(x)}\phi \tag{68}$$

we must introduce a covariant derivative D_μ exactly in the same way as we did when we obtained the QED Lagrangian,

$$D_\mu = \partial_\mu + ieA_\mu\,. \tag{69}$$

Replacing $\partial_\mu \to D_\mu$, the Lagrangian becomes

$$\mathcal{L} = D_\mu\phi^* D^\mu\phi - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{70}$$

where we already added the kinetic term for the gauge field A_μ. $V(\phi)$ is the same as in Eq. (56). Therefore, for $\mu^2 < 0$ the minimum of the potential is not found at $\langle\phi\rangle = 0$, but at $\langle|\phi|\rangle = v = \sqrt{\frac{-\mu^2}{\lambda}}$. We could now split

ϕ into its real and imaginary parts and proceed similarly, applying the shift $\phi = \phi_1^0 + v$ to introduce physical fields that allow to define a proper QFT based on perturbations around the ground state. However, it proves more convenient (the resulting expressions are more transparent) to parameterize ϕ as

$$\phi = \frac{1}{\sqrt{2}} \left(\phi_1^0 + v \right) \exp \left(i \, \frac{\phi_2^0}{v} \right) . \tag{71}$$

ϕ_1^0 represents the modulus of ϕ, already shifted with respect to the chosen minimum, and ϕ_2^0 represents the phase, properly normalized. Plugging this expression into $\mathcal{L}$ one finds

$$\begin{aligned} \mathcal{L} =& \frac{1}{2} \left(\partial_\mu \phi_1^0 \partial^\mu \phi_1^0 + \partial_\mu \phi_2^0 \partial^\mu \phi_2^0 \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left(-2\mu^2 \right) \left(\phi_1^0 \right)^2 + \frac{e^2 v^2}{2} A_\mu A^\mu \\ & + ev A_\mu \partial^\mu \phi_2^0 + \text{interactions} . \end{aligned} \tag{72}$$

This Lagrangian includes a scalar field ϕ_1^0 with mass $m(\phi_1^0) = \sqrt{-2\mu^2}$, a massless scalar ϕ_2^0 (the Goldstone boson) and a massive evector boson A_μ, with mass $m(A) = ev$. However, the presence of the $A_\mu \partial^\mu \phi_2^0$ mixes the A_μ and ϕ_2^0 propagators and complicates the interpretation. We can get rid of this term by making use of the gauge freedom. In order to eliminate it we make the gauge transformation

$$\phi \quad \to \quad \phi' = e^{-i\theta(x)} \phi , \quad \text{with} \, \theta(x) = \frac{1}{v} \phi_2^0(x) , \tag{73}$$

which implies, using the parameterization of ϕ introduced in Eq. (71),

$$\begin{aligned} \phi' =& \exp \left(-i \, \frac{\phi_2^0}{v} \right) \times \frac{1}{\sqrt{2}} \left(\phi_1^0 + v \right) \exp \left(i \, \frac{\phi_2^0}{v} \right) \\ =& \frac{1}{\sqrt{2}} \left(\phi_1^0 + v \right) . \end{aligned} \tag{74}$$

In this particular gauge (called unitary gauge) the Goldstone boson disappears and we get

$$\begin{aligned} \mathcal{L} =& \frac{1}{2} \partial_\mu \phi_1^0 \partial^\mu \phi_1^0 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left(-2\mu^2 \right) \left(\phi_1^0 \right)^2 + \frac{e^2 v^2}{2} A_\mu A^\mu \\ & + \text{interactions} . \end{aligned} \tag{75}$$

As anticipated, the resulting theory constains a massive scalar ϕ_1^0 and, more importantly, a massive gauge boson A_μ. Furthermore, the ϕ_2^0 field, which we identified as the Goldstone boson, has disappeared. This is the Higgs mechanism: the SSB of a gauge theory leads to massive gauge bosons in a consistent and elegant manner.

To better understand where the Goldstone boson has gone we can count the degrees of freedom (d.o.f.) of the theory in the initial and final Lagrangians:

Initial $\mathcal{L}$	
ϕ charged scalar:	2
A_μ massless vector:	2
	4

Final $\mathcal{L}$	
ϕ_1^0 real scalar:	1
A_μ massive vector:	3
	4

As we can see, the d.o.f. of the Goldstone boson has been absorbed by th gauge boson, that acquires a mass. In fact, the Goldstone boson has turned into the longitudinal component of A_μ, the new d.o.f. that has acquired after becoming massive. We say that ϕ_2^0 has been "eaten up" by A_μ.

This mechanism can be easily generalized to non-Abelian gauge theories and is at the heart of the SM, where it is used to give a mass to the W and Z bosons, as we will learn in the next lecture.

2.4 Summary of the lecture

In this lecture we have reviewed the basic ingredients that are required to construct the SM. After discussing the most relevant pre-SM theories of the weak interactions and studying where they failed, we moved on to the discussion of gauge theories (Abelian and non-Abelian) and spontaneous symmetry breaking. With these elements we are now in position to build the SM.

2.5 Exercises

Exercise 1.1 Inverse muon decay ($\nu_\mu e^- \to \mu^- \nu_e$) in the V-A theory. Show that unitarity is violated at $\sqrt{s} \sim 300$ GeV.

Exercise 1.2 Neutrino scattering into longitudinal W-bosons ($\nu_e \overline{\nu}_e \to W_L^+ W_L^-$) in the IVB theory. Consider the high-energy limit and show that it violates unitarity.